Math 112: Introductory Real Analysis

§ Upper and lower limits

Def Let $\{s_n\}$ be a sequence of real numbers.

We write $s_n \to +\infty$ or $\lim_{n\to\infty} s_n = +\infty$ ← both divergent

if for every real M there is an integer N such that

$$s_n > M \text{ for all } n \geq N.$$

Likewise, we write $s_n \to -\infty$ or $\lim_{n\to\infty} s_n = -\infty$

if for every real M there is an integer N such that

$$s_n < M \text{ for all } n \geq N.$$

Def Let $\{s_n\}$ be a sequence of real numbers.

Let $E \subseteq \mathbb{R} \cup \{+\infty, -\infty\}$ be the set of numbers x (in the extended real number system)

such that $s_{n_k} \to x$ for some subsequence $\{s_{n_k}\}$.

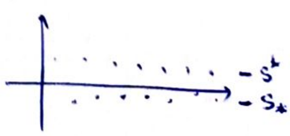
The upper and lower limits of $\{s_n\}$ are defined as

$$s^* := \sup E,$$

$$s_* := \inf E.$$

We also use the notation $\limsup_{n\to\infty} s_n = s^*$, $\liminf_{n\to\infty} s_n = s_*$

Ex Let $s_n = (-1)^n(1+\frac{1}{n})$. Then $\limsup_{n\to\infty} s_n = 1$, $\liminf_{n\to\infty} s_n = -1$.



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<u>Thm</u> Let $\{s_n\}$ be a sequence of real numbers.

Let E and s^* be as before.

Then, (a) $s^* \in E$ ~~(i.e. s^* is a subsequential limit)~~

(b) If $x > s^*$, then there is an integer N such that $s_n < x$ for all $n \geq N$.

(c) s^* is the only number with properties (a) and (b).

Analogous result is true for s_*.

proof)

(a) If $s^* = +\infty$, then E is not bounded above, and hence $\{s_n\}$ is not bounded above. Then there is a subsequence $\{s_{n_k}\}$ such that $s_{n_k} \to +\infty$.

If $s^* \in \mathbb{R}$, then E is bounded above and at least one subsequential limit exists.

<u>The set of subsequential limits form a closed set.</u> so $\sup E = s^* \in E$.

If $s^* = -\infty$, then $E = \{-\infty\}$ and there's no subsequential limit.

(b) Let $x > s^*$. If there were infinitely many values of n such that $s_n \geq x$, then there must be a number $y \in E$ such that $y \geq x > s^*$, contradicting the definition of s^*.

(c) Suppose there were two numbers, p and q, satisfying (a) and (b), and suppose $p < q$. Choose x such that $p < x < q$. Since p satisfies (b), we have $s_n < x$ for all $n \geq N$ (for some integer N), but then q cannot satisfy (a). ∎

Thm If $s_n \leq t_n$ for all $n \geq N$, where N is a fixed integer, then

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n$$

and $$\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n .$$

Some special sequences

The proofs will be based on the following fact: If $0 \leq x_n \leq s_n$ for $n \geq N$, where N is some fixed number, and if $s_n \to 0$, then $x_n \to 0$.

Thm

(a) If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

(b) If $p > 0$, then $\lim_{n\to\infty} p^{\frac{1}{n}} = 1$

(c) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

(d) If $p > 0$ and α is real, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.

(e) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

proof)

(a) Take $N > \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}$. Then $n \geq N \Rightarrow n^p > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n^p} < \varepsilon$.

(b) If $p \geq 1$, put $x_n = p^{\frac{1}{n}} - 1$. Then $x_n \geq 0$, and $1 + nx_n \leq (1+x_n)^n = p$,

so $0 \leq x_n \leq \frac{p-1}{n}$. Hence $x_n \to 0$.

If $p \leq 1$, the result is obtained by taking reciprocals.

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(c) Put $x_n = n^{\frac{1}{n}} - 1$. Then $x_n \geqslant 0$, and, by the binomial thm,

$$n = (1+x_n)^n \geqslant \frac{n(n-1)}{2} x_n^2 .$$

Hence

$$0 \leqslant x_n \leqslant \sqrt{\frac{2}{n-1}} \quad (n \geqslant 2).$$

(d) Let k be an integer such that $k > \max\{\alpha, 0\}$.

For $n > 2k$,

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\cdots(n-k+1)}{k!} p^k > \frac{\left(\frac{n}{2}\right)^k}{k!} p^k .$$

Hence,

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{n^\alpha k!}{\left(\frac{n}{2}\right)^k p^k} = \left(\frac{2}{p}\right)^k k!\, n^{\alpha-k} \quad (n > 2k).$$

Since $\alpha - k < 0$, $n^{\alpha-k} \to 0$ by (a)

(e) Take $\alpha = 0$ in (d)

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